SOLUTIONS TO FINAL EXAM 2025.1.6

PROBLEM 1

(a) DEFINITION OF MANIFOLDS:

A manifold M is a Hausdorff, second countable topological space with a smooth structure $\{(\varphi_\alpha, U_\alpha, V_\alpha)\}_{\alpha\in I}$ s.t.:

(i) $U_\alpha \overset{\text{open}}{\subseteq} M$, $V_\alpha \overset{\text{open}}{\subseteq} \mathbb{R}^n$, $\forall \alpha \in I$;

(ii) $\varphi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism, $\forall \alpha\in I$;

(iii) $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth, $\forall \alpha, \beta \in I$.

(b) Pf. For M^k, N^ℓ smooth mfds, then $M\times N$ is a smooth mfd.

$\Rightarrow$ Smooth constructure $\{(\varphi_\alpha, U_\alpha, V_\alpha)\}_{\alpha\in I}$, $\{(\psi_\beta, X_\beta, Y_\beta)\}_{\beta\in J}$ of M, N respectively, then the smooth structure of $M\times N$ is given by

$$\{(\varphi_\alpha \times \psi_\beta, U_\alpha \times X_\beta, V_\alpha \times Y_\beta)\}_{\alpha\in I, \beta\in J}$$

Easy to check: $U_\alpha \times X_\beta \overset{\text{open}}{\subseteq} M \times N$, $V_\alpha \times Y_\beta \overset{\text{open}}{\subseteq} \mathbb{R}^{k+l}$;

$$\varphi_\alpha \times \psi_\beta : U_\alpha \times X_\beta \to V_\alpha \times Y_\beta,$$

$$(x, y) \mapsto (\varphi_\alpha(x), \psi_\beta(y))$$

is a homeomorphism;

$$(\varphi_\alpha \times \psi_\beta) \circ (\varphi_\gamma \times \psi_\delta)^{-1}$$

$$= (\varphi_\alpha \circ \varphi_\gamma^{-1}) \times (\psi_\beta \circ \psi_\delta^{-1}) \text{ is smooth.}$$

So $M \times N$ is a smooth mfd. #

(c) <u>Sol</u> $S^3 \times \mathbb{RP}^2$ is not orientable.

S^3 closed, $\mathbb{RP}^2$ closed $\Rightarrow S^3 \times \mathbb{RP}^2$ closed.

KÜNNETH $\Rightarrow H^5_{dR}(S^3 \times \mathbb{RP}^2; \mathbb{R})$

$$\simeq H^3_{dR}(S^3; \mathbb{R}) \otimes H^2_{dR}(\mathbb{RP}^2; \mathbb{R})$$

$$\simeq \mathbb{R} \otimes 0 = 0$$

so it is not orientable.

PROBLEM 2

Ans. (a) YES

(b) NO

(c) NO

(d) NO

(e) NO

DETAILS OF ANS

(a) M^n orientable $\Longleftrightarrow$ $\exists$ volume form in $\Omega^n(M)$.

left invariant fields $\{X_i\}_{i=1}^n \subset \Gamma(TG)$

$\rightsquigarrow$ left invariant 1-forms $\{\omega^i\}_{i=1}^n \subset \Omega^1(G)$

volume form: $\omega^1 \wedge \cdots \wedge \omega^n$, $n = \dim G$.

(b) Need the condition that Lie groups are simple connected.

COUNTEREXAMPLE: $\pi: Spin(n) \to SO(n)$ double cover

or $\pi: \mathbb{R}^n \to \mathbb{T}^n$

(c) $f: S^3 \to \mathbb{R}^3$ embedding $\Rightarrow f(S^3) \underset{\text{closed}}{\overset{\text{open}}{\subseteq}} \mathbb{R}^3 \Rightarrow f(S^3) = \mathbb{R}^3$ & cpt.

(d) COUNTEREXAMPLE:

Let $\varphi : \mathbb{R} \to \mathbb{R}$ a bump function, i.e. $\varphi \in C_c^\infty(\mathbb{R})$, $0 \leq \varphi \leq 1$ & $\operatorname{supp} \varphi \subset [-1,1]$ & $\varphi|_{[-\frac{1}{2},\frac{1}{2}]} \equiv 1$.

Let $\{q_n\}_{n=1}^{+\infty} = \mathbb{Q}$, & $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) := \sum_{n=1}^{+\infty} q_n \varphi(x - 2n),$$

so $\operatorname{crit}(f) \supset \bigcup_{n=1}^{+\infty} \left(2n - \frac{1}{2}, 2n + \frac{1}{2}\right) = \coprod_{n=1}^{+\infty} \left(2n - \frac{1}{2}, 2n + \frac{1}{2}\right)$

$f(\operatorname{crit}(f)) \supset \coprod_{n=1}^{+\infty} f\left(2n - \frac{1}{2}, 2n + \frac{1}{2}\right) = \mathbb{Q}$.

(e) Only S^1, S^3, S^7 has trival tangent bundle.

(Due to Lie group structure)

S^{2n} doesn't admit a no-where vanishing vector field.

PROBLEM 3

(a) CONSTANT RANK THEOREM:

Let $f: M^m \to N^n$ be a smooth map s.t. near $p \in M$, rank $f = r$ is constant. Then $\exists$ local charts (φ, U, V) near p & (ψ, X, Y) near $f(p)$ s.t.

$$\psi \circ f \circ \varphi^{-1} : \underset{\mathbb{R}^m}{\overset{}{V}} \to \underset{\mathbb{R}^n}{\overset{}{Y}}, \quad (x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_r, 0, \cdots, 0).$$

$V \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$

(b) STOKE'S THEOREM:

Let M^n be an oriented cpt mfd with boundary, then $\forall\, \theta \in \Omega^{n-1}(\partial M)$, $\int_{\partial M} i^*\theta = \int_M d\theta$, where $i: \partial M \to M$ inclusion.

(c) ADO-IWASAWA'S THEOREM:

Every finite-dimensional real Lie algebra $\mathfrak{g}$ admits a faithful finite-dimensional representation, i.e. $\exists\, n$ s.t. $\mathfrak{g} \xrightarrow{f} \mathfrak{gl}(n; \mathbb{R})$.

(d) HODGE'S THEOREM:

Let M be a closed mfd, $\mathcal{H}^k(M) := \{\alpha \in \Omega^k(M) : \Delta\alpha = 0\}$.

(i) $\mathcal{H}^k(M)$ is a finite-dimensional space over $\mathbb{R}$;

(ii) $\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))$

$\oplus \mathrm{Im}(\delta : \Omega^{k+1}(M) \to \Omega^k(M))$;

PROBLEM 4.

(a) MAYER-VIETORIS SEQUENCE:

$$\cdots \to H^k_{dR}(M;\mathbb{R}) \xrightarrow{\alpha_k} H^k_{dR}(U;\mathbb{R}) \oplus H^k_{dR}(V;\mathbb{R})$$
$$\xrightarrow{\beta_k} H^k_{dR}(U\cap V;\mathbb{R}) \xrightarrow{\delta_k} H^{k+1}_{dR}(M;\mathbb{R}) \to \cdots$$

(b) Pf. (Only consider H^i_{dR} is finitely generated)

$$H^k_{dR}(M;\mathbb{R}) \simeq \mathrm{Im}\,\alpha_k \oplus \ker\alpha_k \simeq \ker\beta_k \oplus \ker\alpha_k,$$
$$H^k_{dR}(U;\mathbb{R}) \oplus H^k_{dR}(V;\mathbb{R}) \simeq \mathrm{Im}\,\beta_k \oplus \ker\beta_k \simeq \ker\delta_k \oplus \ker\beta_k,$$
$$H^k_{dR}(U\cap V;\mathbb{R}) \simeq \mathrm{Im}\,\delta_k \oplus \ker\delta_k \simeq \ker\alpha_{k+1} \oplus \ker\delta_k,$$

$\Rightarrow$ Let $A_k := \dim\ker\alpha_k$, $B_k := \dim\ker\beta_k$, $C_k := \dim\ker\delta_k$,

then: $b_k(M) = A_k + B_k$, $\quad$ $A_0 = 0$

$b_k(U) + b_k(V) = B_k + C_k$, $\quad$ (def of α_0)

$b_k(U\cap V) = A_{k+1} + C_k$, $\quad$ $A_{m+1} = 0$

$$\chi(M) = \sum_{k=0}^{m} (-1)^k b_k(M) = \sum_{k=0}^{m} (-1)^k (A_k + B_k)$$
$$= \sum_{k=0}^{m} (-1)^k (B_k + C_k) + \sum_{k=0}^{m} (-1)^k (A_k - C_k)$$
$$= \chi(U) + \chi(V) - \sum_{k=0}^{m-1} (-1)^k A_{k+1} - \sum_{k=0}^{m} C_k$$
$$= \chi(U) + \chi(V) - \chi(U\cap V).$$

(c) Pf. Choose $S^2 = U \cup V$ such that $U, V \simeq \mathbb{R}^2$, $U, V \overset{\text{open}}{\subseteq} S^2$

$U \cap V$ is homotopic to S^1.

So: $$\chi(M\times S^2) = \chi(\underbrace{M\times U}) + \chi(\underbrace{M\times V}) - \chi(\underbrace{M\times(U\cap V)})$$

both $\simeq M\times\mathbb{R}^2$ homotopic to M $\qquad \simeq M\times S^1$

$$= 2\chi(M) - \chi(M\times S^1)$$

Again, choose $S^1 = X \cup Y$ such that $X, Y \simeq \mathbb{R}$,

$X, Y \overset{\text{open}}{\subseteq} S^1$, $X \cap Y$ is homotopic to $\{a, b\}$.

So: $$\chi(M\times S^1) = \chi(\underbrace{M\times X}) + \chi(\underbrace{M\times Y}) - \chi(\underbrace{M\times(X\cap Y)})$$

both $\simeq M\times\mathbb{R}$ homotopic to M $\qquad$ homotopic to $M \sqcup M$

$$= 2\chi(M) - \chi(\underbrace{M\sqcup M}) = 0$$

$\Rightarrow H^k_{dR}(M\sqcup M;\mathbb{R}) \simeq H^k_{dR}(M;\mathbb{R}) \oplus H^k_{dR}(M;\mathbb{R})$

by M-V seq, so $b_k(M\sqcup M) = 2b_k(M)$

& $\chi(M\sqcup M) = \sum_{k=0}^{m} (-1)^k b_k(M\sqcup M)$

$= 2\chi(M)$.

Finally, $\chi(M\times S^2) = 2\chi(M) - \chi(M\times S^1) = 2\chi(M)$. $\quad$ #

<u>Rmk</u> For (c), one can also use Künneth Formula.

When Künneth Formula holds, i.e.

$$H^k_{dR}(M\times N;\mathbb{R}) \simeq \bigoplus_{p+q=k} H^p_{dR}(M;\mathbb{R}) \otimes H^q_{dR}(N;\mathbb{R}),$$

so $b_k(M\times N) = \sum_{p=0}^{k} b_p(M)\, b_{k-p}(N)$, thus:

$$\boxed{\begin{array}{l} m = \dim M \\ n = \dim N \end{array}}$$

$$\begin{aligned}
\chi(M\times N) &= \sum_{k=0}^{m+n} (-1)^k b_k(M\times N) \\
&= \sum_{k=0}^{m+n} (-1)^k \sum_{p=0}^{k} b_p(M)\, b_{k-p}(N) \\
&= \sum_{k=0}^{m+n} \sum_{p=0}^{k} \left[(-1)^p b_p(M) \cdot (-1)^{k-p} b_{k-p}(N) \right] \\
&= \left(\sum_{k=0}^{m} (-1)^k \chi(M) \right) \left(\sum_{p=0}^{n} (-1)^p \chi(N) \right) \\
&= \chi(M)\cdot \chi(N).
\end{aligned}$$

Then $\chi(M\times S^2) \simeq \chi(M)\cdot \chi(S^2) = 2\chi(M)$.

PROBLEM 5

(a) DEFINITION OF deg

$F: M \to N$ is proper, it induces $F^*: H_c^n(N; \mathbb{R}) \to H_c^n(M; \mathbb{R})$.

According to POINCARÉ'S DUALITY, $\left.\begin{matrix} H_c^n(M;\mathbb{R}) \\ H_c^n(N;\mathbb{R}) \end{matrix}\right\} \simeq \mathbb{R}$.

Let $[\alpha]$ be a generator of $H_c^n(N; \mathbb{R})$, then $\exists \lambda \in \mathbb{R}$ s.t.

$$\int_M F^*\alpha = \lambda \int_N \alpha,$$

$\deg(F) := \lambda$ doesn't rely on the choice of generator.

(b) Sol Recall: represent $\mathbb{S}^1$ as $\theta \mapsto e^{i\theta}$

$d\theta$ is a well-defined volume form on $\mathbb{S}^1$.

Now represent $\mathbb{T}^2$ as $(\theta_1, \theta_2) \mapsto (e^{i\theta_1}, e^{i\theta_2})$, $d\theta_1, d\theta_2 \in \Omega^1(\mathbb{T}^2)$, $d\theta_1 \wedge d\theta_2$ is a volume form of $\mathbb{T}^2$.

$F(z, w) = (w, \bar{z})$, so $F(e^{i\theta_1}, e^{i\theta_2}) = (e^{i\theta_2}, e^{-i\theta_1})$

$\Rightarrow F^* d\theta_1 = d\theta_2$, $F^* d\theta_2 = -d\theta_1$.

$F^*(d\theta_1 \wedge d\theta_2) = d\theta_2 \wedge (-d\theta_1) = d\theta_1 \wedge d\theta_2$

$\Rightarrow \deg(F) = 1$.

(c) <u>Pf</u> Let $f: S^n \to S^n$, $x \mapsto -x$.

<u>Claim</u>: $\deg(f) = (-1)^{n+1}$.

<u>Pf of Claim</u>: Let $i: S^n \hookrightarrow \mathbb{R}^{n+1}$ be the embedding.

Take $\omega \in \Omega^n(\mathbb{R}^{n+1})$ as:

$$\omega = \sum_{i=1}^{n+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1},$$

then $i^*\omega \in \Omega^n(S^n)$ is a volume form.

Let $\tilde{f}: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $y \mapsto -y$, then $i \circ f = \tilde{f} \circ i$.

$$\Rightarrow f^* i^* \omega = (i \circ f)^* \omega = (\tilde{f} \circ i)^* \omega = i^* \tilde{f}^* \omega$$

$$= i^*((-1)^{n+1}\omega) = (-1)^{n+1} i^* \omega. \qquad \#$$

F doesn't have fixed points, then

$$H: S^n \times [0,1] \to S^n, \ (x,t) \mapsto \frac{(1-t)F(x) - tx}{\|(1-t)F(x) - tx\|}$$

is well-defined. (If $\exists (x,t)$ $(1-t)F(x) = tx$, then $t \neq 0, t \neq 1$.

$F(x) = \frac{t}{1-t} x$, $\|F(x)\| = 1 = \|\frac{t}{1-t} x\| = \frac{t}{1-t} \Rightarrow t = \frac{1}{2} \Rightarrow F(x) = x$.)

So H is a homotopy from F to f.

$\Rightarrow \deg(F) = \deg(f) = (-1)^{n+1}$. $\qquad \#$

PROBLEM 6

(a) DEFINITION OF EXPONENTIAL MAP

$\exp: \mathfrak{g} \to G$ is defined by $v \mapsto \varphi^{1}_{X^v}(e)$

where: e is the identity of G, X^v is the left invariant vector field generated by v, $\varphi^{t}_{X^v}$ is the flow generated by X^v.

(b) Pf. Claim $\forall t, s \in \mathbb{R}$, $\exp(tX)\exp(sX) = \exp((t+s)X)$

Pf of Claim. Let $\gamma(s) = \exp(tX)\exp(sX)$, $\tilde{X}$ be the left-invariant vector field generated by X,

then: $\gamma'(s) = dL_{\exp(tX)}\left(\frac{d}{ds}\exp(sX)\right)$

$$= dL_{\exp(tX)}\left(\tilde{X}|_{\exp(sX)}\right) = \tilde{X}|_{\gamma(s)}$$

$$\Rightarrow \gamma(s) = \varphi^{s}_{\tilde{X}}(\gamma(0)) = \varphi^{s}_{\tilde{X}}(\exp(tX)) = \varphi^{s}_{\tilde{X}} \circ \varphi^{t}_{\tilde{X}}(e)$$

$$= \varphi^{s+t}_{\tilde{X}}(e) = \exp((s+t)X) \qquad \#$$

$$\exp(X)^n = \exp(X)^2 \exp(X)^{n-2} = \exp(2X)\exp(X)^{n-2}$$

$$= \exp(2X)\exp(X)\exp(X)^{n-3} = \exp(3X)\exp(X)^{n-3}$$

$$= \cdots = \exp((n-1)X)\exp(X) = \exp(nX). \qquad \#$$

(c) Pf. Recall: BAKER–CAMPBELL–HAUSDORFF FORMULA

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + t^2 Z(t))$$

where $X, Y \in \mathfrak{g}$, t is small enough.

$$\exp(\tfrac{t}{n}X)\cdot\exp(\tfrac{t}{n}Y) = \exp(\tfrac{t}{n}(X+Y) + \tfrac{t^2}{n^2} Z(\tfrac{t}{n})),$$

so $\left(\exp(\tfrac{t}{n}X)\cdot\exp(\tfrac{t}{n}Y)\right)^n$

$$= \left(\exp(\tfrac{t}{n}(X+Y) + \tfrac{t^2}{n^2} Z(\tfrac{t}{n})\right)^n$$

$$\overset{(b)}{=} \exp\left(n\cdot\tfrac{t}{n}(X+Y) + n\cdot\tfrac{t^2}{n^2} Z(\tfrac{t}{n})\right)$$

$$= \exp\left(t(X+Y) + \tfrac{t^2}{n} Z(\tfrac{t}{n})\right)$$

$n \to +\infty$, $\frac{t^2}{n} Z(\frac{t}{n}) = O(\frac{1}{n}) \to 0$.

Thus $\lim\limits_{n\to+\infty}\left(\exp(\tfrac{t}{n}X)\cdot\exp(\tfrac{t}{n}Y)\right)^n = \exp(t(X+Y))$.

#

PROBLEM 7

(a) Take $F: \mathbb{R} \to \mathbb{R}^2$ as $F(x) = (x^2, x^3)$. It's injective.

$$\mathrm{Jac}(F)(x) = \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix}$$ degenerate at $x = 0 \Rightarrow$ NOT immersion.

(b) Take $F: \mathbb{R} \to \mathbb{R}$ as $F(x) = x^3$. It's injective.

$\mathrm{Jac}(F)(x) = F'(x) = 3x^2$ degenerate at $x = 0$

$\Rightarrow$ NOT submersion.

(c) Take $F: \mathbb{R} \to \mathbb{R}^2$ as $F(t) = (\cos t, \sin t)$ NOT injective.

$$\mathrm{Jac}(F)(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$ NOT degenerate everywhere.

(d) Take $F: \mathbb{R} \to \mathbb{R}$ as $F(x) = \arctan x$ NOT surjective.

$\mathrm{Jac}(F)(x) = F'(x) = \frac{1}{1+x^2}$ NOT degenerate everywhere.

PROBLEM 8

(a) <u>Sol</u> F is not-degenerate at $(0,0)$.

$$\operatorname{Hess} F(x,y) = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}, \quad \operatorname{Hess} G(x,y) = \begin{pmatrix} 2y^2 & 4xy \\ 4xy & 2x^2 \end{pmatrix}$$

so $\operatorname{Hess} F(0,0)$ is non-degenerate,

$\operatorname{Hess} G(0,0) = 0$ is degenerate.

(b) <u>Pf</u>. Only to consider the case $f : U(\subset \mathbb{R}^n) \to \mathbb{R}$, since M is locally Euclidean.

Let $x_0 \in U$ be a non-degenerate critical pt of f,

consider $\tau f : U \to \mathbb{R}^n$, $x \mapsto \nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)(x)$,

then: $\operatorname{Jac}(\tau f)(x) = \left(\frac{\partial}{\partial x_j}(\tau f)_i\right)_{i,j}(x)$

$= \operatorname{Hess} f(x)$,

$\det \operatorname{Hess} f(x_0) \neq 0 \overset{\text{IFT}}{\Rightarrow} \exists$ nbhd $\tilde{U} \subset U$ of x_0, nbhd $\tilde{V} \subseteq \mathbb{R}^n$ of $\tau f(x_0) = 0$ s.t. $\tau f : \tilde{U} \to \tilde{V}$ is a diffeomorphism.

$\Rightarrow f$ doesn't have critical pt on $\tilde{U} \setminus \{x_0\}$

since $\tau f \neq 0$ on $\tilde{U} \setminus \{x_0\}$. #

(c) <u>Pf</u> Let $C := \tau F(\mathrm{crit}(\tau F)) \xRightarrow{\text{Morse-Sard}} \mathcal{L}^n(C) = 0$

Take $a \in \mathbb{R}^n \setminus C$, $dF_a(x) = dF(x) - a = \tau F(x) - a$,

$\mathrm{Hess}\, F_a(x) = \mathrm{Hess}\, F(x) = \mathrm{Jac}(\tau F)(x)$.

<u>Case 1</u> $a \notin \mathrm{Im}(\tau F)$. $\Rightarrow \mathrm{crit}(F_a) = \phi$.

If $\exists x_0 \in \mathrm{crit}(F_a)$, then $dF_a(x_0) = 0$

i.e. $a = \tau F(x_0) \in \mathrm{Im}(\tau F)$.

<u>Case 2</u> $a \in \mathrm{Im}(\tau F)$.

For each $x \in (\tau F)^{-1}(a)$, x is a critical pt of dF_a.

$a \in \mathbb{R}^n \setminus C \Rightarrow \mathrm{Jac}(\tau F)(x)$ is non-degenerate

$\Rightarrow \mathrm{Hess}\, F_a(x)$ is non-degenerate

$\Rightarrow F_a$ is always a Morse function when $a \in \mathbb{R}^n \setminus C$.

i.e. a.e. $a \in \mathbb{R}^n$, F_a is a Morse function. #

PROBLEM 9

Pf (a) See HW5 Ex4.

(b) As $\dim \Lambda^{n-k}V^* = \binom{2n}{n-k} = \binom{2n}{n+k} = \dim \Lambda^{n+k}V^*$, by standard linear algebra result, $\wedge\omega^k : \Lambda^{n-k}V^* \to \Lambda^{n+k}V^*$ is injective $\Rightarrow$ $\wedge\omega^k$ is surjective.

(c) Suppose: $\alpha\wedge\omega^k = 0$ for $\alpha \in \Lambda^{n-k}V^* \Rightarrow \alpha = 0$.

Now assume $\alpha \in \Lambda^{n-k+1}V^*$ with $\alpha\wedge\omega^{k-1} = 0$. Then $\alpha\wedge\omega^k = 0$. Take $v \in V$ and:

$$
\begin{aligned}
0 &= \iota_v(\alpha\wedge\omega^k) \\
&= (\iota_v\alpha)\wedge\omega^k - (-1)^{n-k} k\alpha\wedge(\iota_v\omega)\wedge\omega^{k-1} \\
&= (\iota_v\alpha)\wedge\omega^k + (-1)^{n-k} k\underbrace{\alpha\wedge\omega^{k-1}}_{\text{assumption}=0}\wedge(\iota_v\omega) \\
&= (\iota_v\alpha)\wedge\omega^k
\end{aligned}
$$

Then $\iota_v\alpha = 0$. Since $v \in V$ is arbitrary, then $\alpha = 0$.

#

PROBLEM 10

(a) FROBENIUS INTEGRABILITY THEOREM:

Vector fields: Let D^k be a distribution on mfld M^n.

Then D^k is integrable (i.e. $\forall p \in M, \exists N \subset M$ with $p \in N$ s.t. $T_x N = D^k(x), \forall x \in N$)

$\iff D^k$ is involutive (i.e. $\forall X, Y \in D^k$, $[X, Y] \in D^k$).

Differential forms: $I(D^k) = \{\alpha \in \Omega^i(M) : \alpha|_{D^k} = 0\}$

Then D^k is integrable

$\iff I(D^k)$ is a differential ideal (i.e. $dI(D^k) \subset I(D^k)$).

(b) Pf. Let $X = \cos r \frac{\partial}{\partial \theta} - r \sin r \frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial r}$. Then:

$$[X, Y] = \sin r \frac{\partial}{\partial \theta} + (\sin r + r\cos r)\frac{\partial}{\partial z}.$$

Assume D^2 is integrable at (r_0, θ_0, z_0), then $[X, Y] \in D^2$,

$\exists a, b \in \mathbb{R}$ s.t. $[X, Y] = aX + bY$. Then

$$b = 0 \ \& \ \begin{cases} a\cos r_0 = \sin r_0 \\ a r_0 \sin r_0 = -\sin r_0 - r_0 \cos r_0 \end{cases} \qquad (\star)$$

LINEAR ALGEBRA: such $a \in \mathbb{R}$ exists $\Longleftrightarrow$

$$\det\begin{pmatrix} \cos r_0 & -\sin r_0 \\ r_0 \sin r_0 & \sin r_0 + r_0 \cos r_0 \end{pmatrix} = r_0 + \sin r_0 \cos r_0 = 0$$

i.e. $2r_0 + \sin 2r_0 = 0$, the only solution is $r_0 = 0$.

So the integral submfd of $D^2 \subset \{(r,\theta,y) : r=0\}$

$\Rightarrow \frac{\partial}{\partial r} \in (D^2)^\perp$. However, $\frac{\partial}{\partial r} \in D^2$, contradiction!

So D^2 is no-where integrable. #

Rmk Another pf using differential forms of (b):

First determine $I(D^2)$. Still use notations above.

(i) $I(D^2) \cap \Omega^1(\mathbb{R}^3)$.

$I(D^2) \cap \Omega^1(\mathbb{R}^3) = \{\alpha \in \Omega^1(\mathbb{R}^3) : \alpha(X)=0, \alpha(Y)=0\}$.

So: $\alpha(Y)=0 \Rightarrow \alpha$ is spaned by $d\theta$, dz;

$\alpha(X)=0 \Rightarrow \alpha$ is spaned by $r\sin r\, d\theta + \cos r\, dz := \omega$;

so: $I(D^2) \cap \Omega^1(\mathbb{R}^3) = \mathrm{span}_{C^\infty}\{\omega\}$.

(ii) $I(D^2) \cap \Omega^2(\mathbb{R}^3)$.

$I(D^2) \cap \Omega^2(\mathbb{R}^3) = \{\alpha \in \Omega^2(\mathbb{R}^3) : \alpha(X,Y)=0\}$.

If $\alpha(X,Y)=0$, i.e.:

$$\cos r\cdot\alpha\left(\frac{\partial}{\partial\theta},\frac{\partial}{\partial r}\right)-r\sin r\cdot\alpha\left(\frac{\partial}{\partial z},\frac{\partial}{\partial r}\right)=0.$$

So α is spaned by $d\theta\wedge dz$, $r\sin r\,dr\wedge d\theta+\cos r\,dr\wedge dz :=\eta$.

$\Rightarrow I(D^2)\cap\Omega^2(\mathbb{R}^3)=\mathrm{span}_{C^\infty}\{d\theta\wedge dz,\eta\}$.

(iii) $I(D^2)\cap\Omega^3(\mathbb{R}^3)=\Omega^3(\mathbb{R}^3)$.

So $I(D^2)=\mathrm{span}_{C^\infty}\left(\{\omega, d\theta\wedge dz,\eta\}\cup\Omega^3(\mathbb{R}^3)\right)$.

<u>Note</u> $d\omega=(r\cos r+\sin r)dr\wedge d\theta-\sin r\,dr\wedge dz$

$$d\eta=0.$$

Similar to the proof above, $d\omega\notin I(D^2)$ except for $r=0$, contradiction again. #

(c) <u>Pf</u> Let $X=x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+x(y+1)\frac{\partial}{\partial z}$, $Y=\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}$, then:

$$[X,Y]=\frac{\partial}{\partial z}-\left(\frac{\partial}{\partial x}+(y+1)\frac{\partial}{\partial z}\right)=-\frac{\partial}{\partial x}-y\frac{\partial}{\partial z}=-Y,$$

so $[X,Y]\in D^2$, D^2 is integrable. #